All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

Where relevant, in questions 2-6, you may use without proof general results from the theory of Sturm-Liouville eigenvalue problems, provided that the results used are stated clearly.

(1) Consider the Sturm-Liouville differential operator

$$\mathcal{L} \equiv \frac{1}{w(x)}\left(\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + r(x)\right)$$

for real, continuously differentiable functions $w(x) > 0$, $p(x) > 0$ and $r(x)$.

(a) Write down the inner product $\langle f, g\rangle$, acting on pairs of complex-valued functions $f(x)$, $g(x)$ defined on the real interval $a \le x \le b$, that is associated with the Sturm-Liouville operator above.

(b) Show that under this inner product

$$\langle \mathcal{L}f, g\rangle = \langle f, \mathcal{L}g\rangle + \left[p\left(f\bar{g}' - f'\bar{g}\right)\right]_a^b \qquad (\dagger)$$

where primes denote d/dx, and overbars complex conjugates.

(c) Write down, for a general function $y(x)$ defined on $[a, b]$, the most general possible boundary conditions ($\ddagger$) that, when applied to both $f(x)$ and $g(x)$ at $x = a, b$, cause the boundary term in ($\dagger$) to vanish.

(d) Consider the eigenvalue problem on $a \le x \le b$ defined by

$$\mathcal{L}y = -\lambda y, \qquad \text{subject to boundary conditions } (\ddagger).$$

Show that

(i) The eigenvalues $\{\lambda_k\}$ must be real.
(ii) The corresponding eigenfunctions $\{y_k\}$ must be orthogonal.

(2) (a) The Gamma function $\Gamma(x)$ is defined by the integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

Find a relationship between $\Gamma(x)$ and $\Gamma(x-1)$, and use this relationship to express the product

$$\prod_{j=1}^{k}(c+j) = (c+1)(c+2)\cdots(c+k-1)(c+k)$$

in terms of the Gamma function (c is a real constant).

(b) Consider the differential equation

$$(*) \qquad 2x(x^2-1)y'' + (4x^2-1)y' - 4xy = 0.$$

Seeking series solutions of Frobenius type

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+c} \qquad (a_0 = 1),$$

show that successive coefficients in each series solution are related by

$$a_k = \left(\frac{k+c-3}{k+c-\frac{1}{2}}\right) a_{k-2},$$

and find all possible values of the constant c.

(c) Find the two linearly independent solutions of $(*)$, expressing the coefficients in terms of the Gamma function where necessary.

(d) What is the radius of convergence of each series solution?

(3) (a) Axisymmetric solutions $u(r,\theta)$ of Laplace's equation in spherical geometry satisfy

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = 0.$$

Using the method of separation of variables, show that the general solution of Laplace's equation that is regular at the poles ($\theta = 0, \pi$) can be written

$$u(r,\theta) = \sum_{k=0}^{\infty}\left(A_k r^k + \frac{B_k}{r^{k+1}}\right)\mathrm{P}_k(\cos\theta).$$

[You may state without proof that the only solutions of the equation $(1-z^2)w'' - 2zw' + \nu(\nu+1)w = 0$ which are regular at $z = \pm 1$ occur for $\nu = k$ (k integer) and are the Legendre polynomials $\mathrm{P}_k(z)$.]

(b) A large (effectively infinite) perfectly conducting metal block initially has temperature $u = 0$. In the centre of the block is a spherical cavity of unit radius, containing an axisymmetric heat source, which for $t > 0$ supplies a continuous heat flux into the metal block so that

$$\frac{\partial u}{\partial r}(1,\theta) = -H_0\sin^2(\theta), \quad (H_0 > 0 \text{ constant}).$$

Find the resulting steady temperature distribution $u(r,\theta)$ in the block outside of the cavity (i.e. in $1 \le r < \infty$, $0 \le \theta \le \pi$).

[Hint: Rodrigues' formula

$$\mathrm{P}_k(x) = \frac{1}{2^k k!}\frac{d^k}{dx^k}(x^2-1)^k,$$

can be used to calculate the first few Legendre polynomials.]

(4) (a) The generating function formula for Bessel functions is

$$\exp\left(\frac{x}{2}\left(t-\frac{1}{t}\right)\right) = \sum_{m=-\infty}^{\infty} t^m J_m(x).$$

Use this to obtain the results

(i)

$$mJ_m(x) = \tfrac{1}{2}x(J_{m-1}(x) + J_{m+1}(x)).$$

(ii)

$$J'_m(x) = \tfrac{1}{2}\left(J_{m-1}(x) - J_{m+1}(x)\right).$$

(iii)

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos\left(x\sin\theta\right)\, d\theta.$$

Hint for (iii): use the substitution $t = \exp(i\theta)$.

(b) You are given that $y(x) = J_0(x)$ is a solution of Bessel's equation with zero index

$$xy'' + y' + xy = 0,$$

where primes denote derivatives with respect to x.

(i) Show that

$$\frac{d}{dx}\left(x^2\left(y'\right)^2\right) + x^2\frac{d}{dx}\left(y^2\right) = 0.$$

(ii) Use the result in (i) to show that

$$\int_0^{j_{0n}} x\left(J_0(x)\right)^2\, dx = \tfrac{1}{2}j_{0n}^2\left(J_0'(j_{0n})\right)^2,$$

where j_{0n} denotes the nth zero of $J_0(x)$ (i.e. $J_0(j_{0n}) = 0$).

(5) A real function $f(x)$ and its Fourier transform $\hat{f}(k)$ are related through

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, \mathrm{d}x \quad \text{and} \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} \, \mathrm{d}k.$$

(a) Find the Fourier transform of the function

$$F(x) = \begin{cases} x & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}.$$

(b) Find the Fourier transform of

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y) g(y) \, \mathrm{d}y,$$

where $f(x)$ and $g(x)$ are functions in $L^2(\mathbb{R})$.

(c) Use Fourier transforms to find the solution $f(x)$ to the integral equation

$$\int_{-\infty}^{\infty} f(x-y) \exp\left(-\frac{y^2}{a^2}\right) \mathrm{d}y = \exp\left(-\frac{x^2}{b^2}\right), \qquad (a, b \in \mathbb{R}, \quad 0 < a < b).$$

You may quote the result

$$\int_{-\infty}^{\infty} \cos(qx) \, \exp\left(-\frac{x^2}{c^2}\right) \mathrm{d}x = c\sqrt{\pi} \exp\left(-\frac{q^2 c^2}{4}\right),$$

but must prove all other results that you use.

(6) A function $f(t)$ defined on $[0, \infty)$ has a Laplace transform $\mathcal{L}[f](s) = \bar{f}(s)$ defined by

$$\bar{f}(s) = \int_0^{\infty} f(t) e^{-st} \, \mathrm{d}t.$$

(a) Find the Laplace transforms of the following functions

$$\text{(i)} \quad f_1(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & t \geq 1 \end{cases} \quad \text{(ii)} \quad f_2(t) = \ddot{f}(t) \quad \text{(ii)} \quad f_3(t) = \cos(\alpha t).$$

where α is a real constant and $\ddot{f}(t)$ denotes the second derivative of $f(t)$.

(b) Write down a formula for the inverse Laplace transform. Take particular care to define the path of integration in the complex s-plane.

(c) Using any method at your disposal, find functions $g_1(t)$, $g_2(t)$ and $g_3(t)$ that have the following Laplace transforms

$$\text{(i)} \quad \bar{g}_1(s) = \frac{1}{s+a}, \quad \text{(ii)} \quad \bar{g}_2(s) = \frac{1}{s^n}, \quad \text{(iii)} \quad \bar{g}_3(s) = \frac{e^{-qs}}{s^2 + b^2},$$

where a, b and $q > 0$ are real constants and $n \geq 1$ is an integer.

END OF PAPER